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# Equivalent measures on product spaces

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Let $(\Omega_n, \mathcal{F}_n, P_n)$ be a probability measure space for each $n \geq 1$ and let $(\Omega, \mathcal{F}, P) = \prod_{n=1}^{\infty} (\Omega_n, \mathcal{F}_n, P_n)$ be the product probability measure space. Assume that for each $n \geq 1$ there exists a $\sigma$-finite measure $\mu_n$ defined on the $\sigma$-algebra $\mathcal{F}_n$.

A $\sigma$-finite measure $\mu$ defined on the $\sigma$-algebra $\mathcal{F}$ is said to be a $\sigma$-finite quasi-product measure of $\{\mu_n\}_{n\geq 1}$ if for each $n \geq 1$

$$\mu = \prod_{i=1}^{n} \mu_i \times \mu_{n+1}^{*}$$

where $\mu_{n+1}^{*}$ is a $\sigma$-finite measure defined on the product space $\prod_{i=n+1}^{\infty} (\Omega_i, \mathcal{F}_i)$.

Now the problem is that ; Under what conditions on $\{P_n\}_{n\geq 1}$ and $\{\mu_n\}_{n\geq 1}$ does there exist a $\sigma$-finite, quasi-product measure $\mu$ equivalent with a product probability measure $P$? (The equivalence of measures means that $P(A) = 0$ iff $\mu(A) = 0$). This problem was first discussed by S. Kakutani who gave a necessary sufficient condition for the existence of an equivalent, finite, quasi-product measure $\mu$, where $\mu(\Omega) = \mu_n(\Omega_n) = 1$ $n \geq 1$. The $\sigma$-finite case was discussed by C. C. Moore and O. Takenouchi when each $\Omega_n$ is a finite or countable set and by D. Hill when $\Omega_n$ is a general probability measure space. Quasi-product measures appear in the existence problem of invariant measures under adding machine transformations acting on infinite product space and in the classification of infinite tensor product factors.

Our purpose is to systematically study the existence problem and to obtain a more simple and efficient condition for the existence of $\sigma$ -finite, quasi-product measures.

(1) There exists a $\sigma$-finite, quasi-product measure $\mu$ of $\{\mu_n\}_{n\geq 1}$ equivalent with a product probability measure P if and only if there are positive constants $b_1, b_2, \ldots$ such that

$$\prod_{n=1}^{\infty} \frac{X_n(\omega)}{b_n}$$

converges with probability 1, where $X_n(\omega) = \frac{d\mu_n}{dP_n}(\omega_n)$.

Using (1) and the Kolmogorov's three series theorem, we have

(2) Necessary sufficient condition of the existence of $\sigma$-finite quasi-product measure $\mu$ of $\{\mu_n\}_{n\geq 1}$ equivalent with P is that there exists a $\mathcal{F}_n$-measurable set $A_n$ for each $n \geq 1$ such that

$$\sum_{n=1}^{\infty} 1 - \frac{E(\sqrt{X_n} : A_n)^2}{E(X_n : A_n)}$$

converges. This is the D. Hill's condition.

(3) There exists a finite, quasi-product measure $\mu$ of $\{\mu_n\}_{n\geq 1}$ equivalent with P if and only if there are positive constants $b_1, b_2, \ldots$ such that

$$\prod_{n=1}^{\infty} \sqrt{\frac{X_n(\omega)}{b_n}}$$

converges in the $L^2(\Omega, P)$-sense and also if and only if

$$\sum_{n=1}^{\infty} 1 - \frac{E(\sqrt{X_n})^2}{E(X_n)}$$

converges. The last condition is the Kakutani's condition, where $\mu_n(\Omega_n) = 1$.